A RADICAL PHENOMENOLOGY OF MATHEMATICS

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RESUMEN

En este artículo se trata de expresar una nueva visión de las matemáticas, en el sentido filosófico más profundo. Se abordan los principales problemas ontológicos y epistemológicos de las matemáticas, y se brindan opiniones consistentes y totalizantes, que sintetizan una elaboración intelectual de muchos años por parte del autor. Este trabajo constituye una verdadera fenomenología de las matemáticas.

In this paper I would like to suggest a picture of Mathematics. Before anything else, I want to define the basical intellectual premises this picture should be consistent with. We want a view of Mathematics which: 1-establishes the nature of Mathematics connected to the empirical and sensorial realms; 2-understands Mathematics as an active part of natural science, which aims to understand and describe the physical reality; 3-sustains a non-aprioristic view of Mathematics; 4-sustains an epistemology that gives active roles to the subject as well as to the object; 5-attempts to give an answer to the problem of truth in mathematics, consistent with the approach to the same "problem" in science in general; 6-comprehends the existence of abstract and empirical dimensions in Mathematics as parts of one unified reality; 7-understands Mathematics as a human practice that needs to be tracked historically and socially in order to apprehend its nature; 8-draws a connection between ontology, epistemology and the historical evolution of Mathematics; 9-makes possible an understanding of the role of "pure" Mathematics in epistemological and historical terms; 10-analizes Mathematics as a vigorous component of the "deconstruction and reconstruction" of the building of modern science; 11-can suggest an intellectual alternative to Rationalism and Neo-Positivism [1] in the theoretical understanding of the nature of Mathematics, and could give a more elaborated version of the empiricist approaches [2]; 12-avoids "historicist" views of Mathematics which neglect or reduce the existence of sensorial and empirical objects in Mathematics [3], 13-allows the elaboration of a program for research within an interdisciplinary approach, including philosophers, mathematicians, historians, educators, psychologists, and other scientists connected to the practice of Mathematics.

I intend to give a picture but not to prove my view. In that sense, it is only an outline, full of insights, strong opinions, and suggestions more than a collection of highly structured theses.

There are mathematical objects. My point is that they are not mental or merely abstract. And,

[1] See Ayer. 1936.

[2] The approach of Mill was very naive; it is not the case of the recent version presented by Philip Kitcher, see Kitcher 1983 and Kitcher 1988.

[3] I do not agree with Brunshvicg not with certain parts of Kitcher's view. See Brunschvicg 1981 and Kitcher 1983.

on the other hand, I think they are physical, but in a precise way. This arises from my theory of knowledge, which establishes that objects in knowledge are the result of a subject-object epistemological relationship. The subject here is not stated as an intellectual figure but as a physical reality. What I mean is: there is a reality outside the observer, but what we apprehend is a combination of the impacts of the outside-object and the observer. I want to underline the conditions of the observer that (material) give a component of the knowledge object. Furthermore, it is the way we apprehend the external things which gives us the form and the structure of our knowledge. The objects of science are not merely our perceptions, but I believe all the knowledge is conditioned by our perceptions. The laws or realities we apprehend as descriptions of the world are conditioned by our perceptions. Thus, we should talk more properly about perception-objects or perceptible-objects than objects in themselves. Like Kant, I accept the existence of an external reality but I think this reality comes to us through our specific conditions. I do not believe the defining conditions of ourselves are mental but physical in the wider sense of this term. And these conditions are the ones which intervene in the knowledge process. Put in other terms, I suggest the existence of three dimensions in the structure of our knowledge. One is the external and independent-to-the-subject reality; another is the world of human perceptions (which can be extended and modified through tools); and the other dimension is the one which includes mental operations, abstractions, representations. The first two dimensions give us the "ontological" basis for the third one. My aim here is to underline the second dimension as a material basis which defines to a great extent the form of knowledge because it is the one dimension where we include the material conditions of the subject. In general terms, the second dimension is the one which makes the connections between the other dimensions. The second and the third dimensions together are the realm where we integrate the entire subject's action (not only perceptions).

In this wide spectrum we come to define what mathematical objects are. The spectrum varies in terms of the role played by the observer or the external reality. We could set a "scale" to "order" our different knowledge-objects within this framework. In the case of mathematics we deal with different objects, positioned in different places of the scale. It is clear that the objects of mathematics can exist even if humans are not aware of them -but depend on the existence of humans and their conditions. This is what happens with all sciences.[4] The emergence of an object constitutes the beginning of its history (for us). The way how people discovered or approached them, or defined them, is very important. The object is just the base upon which human mind can construct theoretical results. [5] I should say that the "input" brought by the subject in the mathematics is "wider" than in the other sciences, even if there are differences between the other sciences in relation to this. We will explain this later.

In very general terms, we could say that the object of science is the space-time reality. Thus, we should find all science particular objects within that reality. In our approach, space and time are materially and epistemologically understood in the subject-object terms we mentioned earlier. (That is to say, space-time is a concept related to our perceptions and to an external-independent world). Therefore, the space-time reality is the general object of mathematics as it is part of science. But to say that does not gives enough understanding of particular sciences and mathematics itself. Particular features within space-time are the particular objects of

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[4] I do not believe in any kind of Platonism.

[5] Classical Platonist views can be studied in Frege 1984 and in several of the essays included in Gödel 1981.

particular sciences. Now, the way sciences were defined or the way they arose becomes a historical issue. What has been understood by Physics, Biology, Mathematics or Chemistry it has been established socially and historically. Therefore, the borders of the different sciences are due to cultural and historical processes; and the distinctions between sciences have been subjected to difference and to change -within certain evident limits.

I am going to suggest the existence of certain general-objects in the history of mathematics. However, this does not mean that they are the only ones possible. That means I believe that we can become aware of more such wide objects as mathematics and knowledge in general evolutes. I am going to describe them as perception-objects or perceptible objects assuming the theoretical framework defined above. To prevent any source of misunderstanding, the general-objects I will suggest are broad realms where we can somehow integrate the particular objects.

The first general-object is the diversity, the existence of different things. This is something that arises from the fact we recognize that reality, or there is an external reality which impacts on us producing that recognition. If we were strange creatures with a different sensorial perception unabling us to perceive diversity, this would not be a mathematical object -or it would be different. On the other hand, the diversity can be understood in a wide number of ways, or we could say there are different features embodied within that notion and not all of them are related to mathematics. I would say then that the feature we are recalling is the same one that allows us -for example- to perceive that there is more than one thing in the world. It is the perception that there is one or a number of things in abstraction of other sensorial-physical properties. Let us say that there is a property of the world and a set of perceptions which allowed us to conceptualize it. It is clear that the recognition of diversity depends very much on our perceptual capabilities.

The second general-object is related to the former, or we may say is a part of it. Let us call it sequentiality. That is to say, the diversity of time events. We could say that time is continuous but not as a first approach. The first perception is the existence of different events. In my opinion both objects are the basis for natural numbers and Arithmetic.

The third general-object is another reality which arises from the subject-object epistemological relationship: the continuum. This is the continuity of time and space as we perceive it not as it might be outside our reality. It is the same idea I used before; we can think of other completely different beings (able to conceptualize, etc.) not able to perceive the continuity we perceive or perceiving another type of continuity should have another kind of mathematics.

The fourth general-object is related to the former: the infinity. Somehow it exists as a space reality and a time reality. It arises from the extrapolation of events as well as a perception of non-ending processes. Non-ending processes can include human operations. Brought together with continuity this opens the emergence of the Calculus and the Analysis. [6]

The fifth general-object is the perception of randomness. It is the presence of the unpredictable in a process or a collection of events. It is the reality that breaks out determinism in science and in knowledge. This is an object related to Probability. Nevertheless, I think Probability is specially connected to Arithmetic, thus to diversity and sequentiality.

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[6] See Boyer 1968.

The sixth general-object is the realm of space shapes and distances. This is the base-object of Geometry.

Perhaps, to follow a more historical path, we should put the latter general-object with the first two ones.

There are inter-relations between these objects, because they are part of an unified reality subject-object. That means that we find in mathematical theories intermingled base-objects. The interrelations, however, are not the same.

On the other hand, I think these general perception-objects do not have the same status epistemologically. I mean that they are different and they occupy different levels. Upon these objects humans have worked creating a huge building of concepts, notions, theories, etc., that we call mathematics. Even if these are specific objects of mathematics I do not suggest these objects have been separated from the rest of sciences. Perhaps it could be asserted that diversity and sequentiality are more general than the others.

So far, we have suggested that the "object" of mathematics as physical reality can be structured in three dimensions: space-time; general-objects; particular objects. The mathematical practice is carried out upon these objects. How is this done? We can say that there are always present two means in this practice: the mental operations human can do (not only logic), and the continuously present input of a relationship with the outside. These are connected in historical terms. The operations are not done in the vacuum but within precise human communities and precise historical conditions. The relation with the "outside" gives an awareness of the subject-object conditions that is expressed in cultural terms, as knowledge or science. This is interesting: the awareness of the subject-object relation is culture in a broad sense, which integrates almost everything. A good question would be whether there are new objects created by new states of awareness. Could new resulting theories bring out new objects?

But before that, let us formulate more questions: how do we relate to those objects, do we describe their properties, should it be better to describe them as intuitions arisen from the sensorial realm? First of all, I think that mathematical concepts arise from those objects -in general terms- but are all abstractions. That is to say: mental creations. When we say green we are describing a relationship between the external object and the observer. The word "green", or the concept it embodies, arises from such reality but it is an abstraction. The same occurs when we say straight line. Perhaps the difference is that there is more abstraction in the latter. The objects I describe-define above are a generalization or an abstraction of many other objects we can describe as mathematical objects. Say lines, triangles, points, refer to shapes but each one -in this case- has an object from which it arises. (Nevertheless, I do not think it is possible to determine all objects from which mathematics concepts arise, or to think all mathematics concepts and notions have reference objects. Some of the mathematics concepts and notions are mental constructs without connection to the epistemological dimension we are considering here as basing mathematical knowledge).

I am going to define e-distance as the nearness that exists between mathematics concepts and mathematics objects. I suggest that this e-distance has varied throughout the history of mathematics. In the beginning e-distance "tended to zero", but it has become longer. What I am trying to stress here is the abstraction of mathematics has increased, understanding that in the sense of a certain separation from the empirical objects we defined before. However, mathematics has always involved combinations of different concepts with different e-distances. That means that a certain trend of mathematics with e-distance 5 -for example- might find the emergence of a new 2-distance concept which participates within the basically 5 e-distance trend.

Until now we have introduced in mathematics general-objects, particular objects, concepts. Now, we should say a word about methods. The methods of mathematics arise from the mental capabilities of human beings. Abstraction is a word to point out but it is limited. We can say there are many abstraction-types. We can think in terms of operations too. But, perhaps, both terms are not sufficient to describe mathematics construction. The creation of a new concept does not seem to be described properly by the word operation. Operations seem to be related to certain patterns perfectly definable. However, the linguistic problem should not distract us from the epistemological reality, that is to say the existence of a variety of mental methods acting upon the mathematical objects and concepts.

But, to affirm the diversity of methods is not enough. We need to say that these methods have an epistemological basis. The methods we are able to use come from the same epistemological object-subject realm we defined earlier. The foundation is in the physical and social actions humans are able to perform. Physical and social actions without material content are part of those methods (let us say grouping, reverse movements, association, etc.). But that is not enough. The basic input comes from mental representations; something we can define as similar to the creation of images [7] brought to mind from the material epistemological realm. This representation is not a devoid-material-operation. It is different. It is part of the capabilities humans are able to use; its roots are biological and are hereditary. (Here is where I lean a little bit on Piaget's theory).

I would say that logic plays a special role here. I understand logic as a collection of rules used in the thinking process. Somehow logic gives us the rules to pass from a certain mental point to another without losing the connection to the operations that are logic's references. Logic is part of the collection of content-devoided actions. But, perhaps in a precise way. It permits us the passage between different propositions in a valid way; a valid way means that if we give any physical content to the propositions, a logical consequence should bring us to an expected physical reality. The rules of logic are defined or grasped after a historical process of using those rules. These rules of thinking were not used always. They are part of the evolution of the human race. Valid inferences required a long experience in the material subject-object history. They are part of the long process of awareness by the human subject. What I suggest is that the word logic refers to two things. On the one hand, it is one use; and, on the other, a conceptualization of this use. The "correct" rules of logic are connected to the relationship between men and nature and society. The use of logic should be seen as a process of learning by the human being (it took thousands of years). The rules of logic have to be associated with the human adaptation and knowledge of reality. This process has to be understood as achieved in the subject-object realm we defined before. The conceptualization of logic is the theoretical description of the process of use of logic. Thus, logic as a science refers to the understanding and explanation of that use. Thus, the history of logic should not be considered to start with the conceptualization, but with the processes and realities related to the use or rules we associate with logic. Furthermore, logic as a science should not be separated from the

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[7] I understand "image" not only in visual terms; I recall that blind persons have images based upon different perceptive and mental processes.

conditions determined by its use and development. In this sense, it is clear that logic cannot be separated from the human mental actions and from the material realm.

Does any mental process produce mathematics? That is the point where we need to give concrete epistemological definitions. Mathematics mental actions are: on the one hand, representations of material realities; and on the other hand, material content-devoided actions (what we can define as operations). [8] Here, material means for us a subject-object reality and not just an external object-reality. Besides, we should include as a general method the abstraction of mathematical concepts or operations. Perhaps, the best places in mathematics where we can see the presence of the first and the latter mental "actions" are in the processes of mathematical definitions. We can say these three types of mental actions comprehend mathematics methods.

The concrete use of these types of methods on objects and concepts and theories is done in historical terms. The importance or the role of each of these mental actions in the mathematical practice could be tracked socially and historically. That means that the possibility -sometimes acceptance- of this importance or role depends on the social and cultural communities where the practice is developed.

In relation to the "acceptance" of particular methods: sometimes certain operations or representations do not become accepted until they show they fulfill certain conditions mathematicians define or demand. These set of conditions or demands vary through history. Rigor is one of the most important conditions that has varied through all mathematics evolution. [9]

I have said that there is a process of content-devoidance, but I think that here it is possible to find different levels and dimensions. I mean that the devoidance is not necessarily thoroughly completed always. There are degrees of devoidance. Or maybe different types of devoidance processes. Perhaps we can define logic as the bottom point of devoidance. Or -even better- as a type of devoidance-process. Logic in our theory is part of mathematics. In this reasoning we are coming close to the same ideas we suggested with the e-distance.

Mathematics objects constitute an open set. Different objects can arise through the evolution of mathematics. I believe that is true because the relation of humans and nature evolves too. We cannot say the set of perceptions of the world can be closed in a certain moment. And, on the other hand, the action of humans on the world modifies the relationship and creates new material realities. It is clear also that the movement of history and culture gives new conditions to this relationship. When we achieve the use of new techniques we deconstruct the former reality and bring up a different set of conditions not only cultural and mental, but material. In the same way, science and knowledge can effect our material perception bringing up new objects. In both such cases the emergence of mathematical objects is possible in a very wide form. [10]

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[8] These are the means through which are possible "obvious" parts in Mathematics; "obvious" in the sense used by Quine in 1953.

[9] Different but useful historical accounts can be found in Boyer 1968, Kline 1980, Bell 1937, 1940, 1951, and Bourbaki 1974.

[10] See Kline 1959.

New mathematics objects can furnish new mathematics concepts. We can say then that there are 2 worlds, one of the mathematical objects and another of the mathematics concepts or notions which refer to those objects. The dynamics of mathematics is given by this interplay of worlds. We can say both worlds have their own different rules of evolution. But, the world of mathematics concepts evolves above the mathematics objects world, although not in a deterministic way. Particularly, mathematics concepts and theories can bring up new objects. The connection between both worlds is always precise and concrete, and should be able to be studied in historical terms. But always being aware that not all mathematics concepts arise from mathematics objects in our sense; and at the same time understanding that they can play very important roles in the evolution of mathematics. This is because the making of mathematics is always a result of human decisions in individual and collective ways.

But -furthermore- the realm of mathematics objects is the one which defines the practice of mathematics and should condition its place in the broad sense. Mathematics would not exist without the existence of this realm.

Let us answer the questions we raised above. We relate to the mathematical objects through our perception. The relation is understood within the framework of the interplay between the 2 worlds we defined before.

About the issue of mathematics as description of objects: mathematics can be a description of this world of objects. (Understanding that description cannot mean absolute truth or more that an approximation to reality). But other remarks are necessary. The objects of mathematics are -lets say- general in a special way. This conducts us to a reflection on the nature of perception. Human perception involves our physical capabilities of reaction to external stimulus but at the same time a process of "recognition" of the stimulus. If there is an active role of the subject there should be a process of "subjectivizing" of the perception itself. That is to say there are more factors than the mere reception of an external impulse. Somehow, perception involves a process of image-creation. Perhaps, this latter process is the basis of our conceptualization capabilities. This process of image-creation should be found -up to certain degrees or in different forms- in animal perception too. Our biological capabilities for abstraction should describe different image-creation processes and different types of perception. However, what we want to underline here is that perception should be seen less than a pasive stand of the subject and more as an active one. Thus, perception should be understood as a combination of active factors. On the other hand, there are many types of possible perceptions, which bear different characteristics. The study of perception should be pursued in the more specific terms.

Perception is a particular process. Each time we perceive something we are having or performing a unique act. When we see a green tree once, next time we see that same tree the perception will be different. If this is true even with things we can state as "the same", it is "more true" of different physical things. The word green is applied to different things we collect under the green concept. It is clear that we are able to do this abstraction because there are physical conditions allowing us to do it, conditions within ourselves as physical beings, and conditions outside which provoke or participate in the creation of our perceptions. When we say green we are saying an abstraction but at the same time there is a set of particular perceptions - and therefore of an external-object- that can be associated with green.

It is the same with mathematics objects. We have different perceptions of continuity an sequentially. But we include through those words or notions collections of particular perceptions. Thus, mathematics objects as defined here are referred to collections of perceptions (and an external reality) which can be embedded in abstractions we can precisely define.

The building of these collections or the building of the concepts used arises from our physical characteristics. Through human culture's evolution we have been able to use tools which have allowed us to deconstruct and reconstruct these collections of perceptions. The use of the microscope and the telescope brought to us different limits to our perceptions and the configuration of our notions, concepts, etc.

Finally, because of the additional input of the subject in the configuration of mathematics objects, their results can be more general. Let me explain this more. Even if I suggested the objects behind the concept green are the same type as those of mathematics, I should -up to this point- advance my position. If science's epistemological objects are related to collection of particular perceptions, we cannot put different sized or quality collections under the same terms. That is to say that the differences between these collections give us different knowledge trends or sciences. I suggest that overall mathematics corresponds to "big-sized" collections (without forgetting quality differences). I am trying to stress the point that mathematical objects are more general than the objects of other sciences; even if I think sciences have overlapped objects.

With the opinions about what is the nature of mathematics that I have stated here, we can begin to understand this science or this collection of sciences in a radically different form.

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